Dimension-2 poset competition numbers and dimension-2 poset double competition numbers ☆

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ABSTRACT

Let $D = (V(D), A(D))$ be a digraph. The competition graph of D, is the graph with vertex set $V(D)$ and edge set $\{uv \in \binom{V(D)}{2} : \exists w \in V(D), \overrightarrow{uw}, \overrightarrow{vw} \in A(D)\}$. The double competition graph of D, is the graph with vertex set $V(D)$ and edge set $\{uv \in \binom{V(D)}{2} : \exists w_1, w_2 \in V(D), \overrightarrow{uw_1}, \overrightarrow{vw_1}, \overrightarrow{w_2 u}, \overrightarrow{w_2 v} \in A(D)\}$. A poset of dimension at most two is a digraph whose vertices are some points in the Euclidean plane $\mathbb{R}^2$ and there is an arc going from a vertex (x_1, y_1) to a vertex (x_2, y_2) if and only if $x_1 > x_2$ and $y_1 > y_2$. We show that a graph is the competition graph of a poset of dimension at most two if and only if it is an interval graph, at least half of whose maximal cliques are isolated vertices. This answers an open question on the doubly partial order competition number posed by Cho and Kim. We prove that the double competition graph of a poset of dimension at most two must be a trapezoid graph, generalizing a result of Kim, Kim, and Rho. Some connections are also established between the minimum numbers of isolated vertices required to be added to change a given graph into the competition graph, the double competition graph, of a poset and the minimum sizes of certain intersection representations of that graph.

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1. Introduction

A *digraph D* is a pair $(V(D), A(D))$ of sets, where $V(D)$ is the set of *vertices* and $A(D) \subseteq V(D) \times V(D)$ the set of *arcs*. We often write $\overrightarrow{uv}$ for the ordered pair $(u, v) \in V(D) \times V(D)$. A vertex v of D is a *source* if there is no $w \neq v$ satisfying $\overrightarrow{wv} \in A(D)$ and a vertex v of D is a *sink* whenever there is no $w \neq v$ such that $\overrightarrow{vw} \in A(D)$. A vertex which is both a source and a sink is called an *isolated vertex* of the digraph and a vertex which is neither a source nor a sink is called an *ordinary vertex* of the digraph.

A *partially ordered set*, also called a *poset*, as a universally accepted shorthand, is an acyclic transitive digraph. That is, for a poset D, $A(D)$ does not contain any loop $\overrightarrow{uu}$ and $\overrightarrow{uv}, \overrightarrow{vw} \in A(D)$ implies that $\overrightarrow{uw} \in A(D)$. A poset $D = (V, A)$ is also conveniently represented by $D = (V, <)$ where $<$ is the *partial order* for which we have $v < u$ if and only if $\overrightarrow{uv} \in A(D)$. A source in a poset $(V, <)$ is also referred to as a *maximal* element with respect to the partial order $<$ and a sink in $(V, <)$ is usually named as a *minimal* element with respect to the partial order $<$. As a measure of its nonlinearity [25], we define the *dimension* of a poset $(D, <)$ to be the minimum n for which there is a mapping from $V(D)$ to $\mathbb{R}^n$ such that $p < q$ in D if and only if $f(p)$ is less than $f(q)$ componentwise. Posets of dimension at most two have interesting structural properties [1,2,4,59]

and can be recognized in linear time [42]. However, it already becomes NP-complete to recognize posets of dimension three [63], not to mention posets of greater dimensions.

A *graph G* is a symmetric digraph, namely a pair $(V(G), E(G))$ of sets, where $V(G)$ is the set of *vertices* and $E(G) \subseteq \binom{V(G)}{2}$ the set of *edges*. We use the notation uv for an unordered pair $\{u, v\} \in \binom{V(G)}{2}$. An *isolated vertex* of a graph is the vertex which appears in no edges of that graph. For any digraph $D = (V, A)$, its *underlying graph*, which we denote by G_D, has vertex set V and edge set $\left\{ uv \in \binom{V}{2} : \{\overrightarrow{uv}, \overrightarrow{vu}\} \cap A \neq \emptyset \right\}$.

Given a digraph $D = (V(D), A(D))$, the *competition graph* $\mathbb{C}(D)$ of D has the same vertex set as D and has an edge uv if for some vertex $x \in V(D)$ both $\overrightarrow{ux}$ and $\overrightarrow{vx}$ are in $A(D)$. The notion of competition graph was introduced by Cohen [10] in the study of ecological niches and have been explored since then by many due to its wide applications [34,48,51,53]. The *double competition graph* of a digraph D, denoted $\mathbb{DC}(D)$, has the same vertex set as D and $uv \in E(\mathbb{DC}(D))$ if and only if there are $w, x \in V(D)$ such that $\overrightarrow{wu}, \overrightarrow{wv}, \overrightarrow{ux}, \overrightarrow{vx} \in A(D)$. The double competition graph is also known as the *competition-common enemy graph*, which was first investigated by Scott [57] as a variant of the competition graph and has been intensively studied by many others [26,31,40,57,58]. Competition graphs of posets are known as strict lower bound graphs of posets while competition graphs of posets with loops attached at all vertices are known as lower bound graphs of posets. These concepts and similarly defined concepts like upper bound graphs and double bound graphs of posets are well studied in the literature [7,18–20,44,46,47,56]. For a good discussion on competition graphs and related issues, we refer to [45, Chapter 4].

For any positive integer d, the *dimension-d poset competition number* $\mathcal{PK}^d(G)$ of a graph G is defined to be the smallest nonnegative integer p such that G together with p additional isolated vertices is isomorphic to the competition graph of a poset of dimension at most d; if no such p exists, $\mathcal{PK}^d(G)$ is taken to be ∞ (similar conventions are used throughout the paper). The *poset competition number* $\mathcal{PK}(G)$ is the smallest nonnegative integer p such that we can obtain from G the competition graph of a poset by adding p isolated vertices. The *dimension-d poset double competition number* $\mathcal{DPK}^d(G)$ of a graph G is defined to be the smallest nonnegative integer p such that G together with p added isolated vertices is isomorphic to the double competition graph of a poset of dimension at most d. The *poset double competition number* $\mathcal{DPK}(G)$ is the smallest nonnegative integer p such that we can obtain from G the double competition graph of a poset by adding p isolated vertices. This paper is on these various poset (double) competition numbers, especially those for dimension-2 posets, and related topics.

It clearly holds for any graph G that

$$\begin{cases} \mathcal{PK}^1(G) \geq \mathcal{PK}^2(G) \geq \cdots \geq \lim_{t \to \infty} \mathcal{PK}^t(G) = \mathcal{PK}(G), \\ \mathcal{DPK}^1(G) \geq \mathcal{DPK}^2(G) \geq \cdots \geq \lim_{t \to \infty} \mathcal{DPK}^t(G) = \mathcal{DPK}(G). \end{cases} \tag{1}$$

For any $n_1, \ldots, n_t \geq 0$, $K_{n_1, \ldots, n_t}$ stands for the *complete t-partite graph* (V, E) with V being a disjoint union of $V_1, \ldots, V_t$, $|V_i| = n_i, i = 1, \ldots, t$, and $E = \left\{ e \in \binom{V(G)}{2} : |\{i : e \cap V_i \neq \emptyset\}| = 2 \right\}$. We use K_t as an abbreviation for $K_{n_1, \ldots, n_t}$ when $n_1 = \cdots = n_t = 1$ and call it the *complete graph of order t*. K_0 will mean the graph whose vertex set is the empty set and will also be counted as a complete graph.

Both $\mathcal{PK}^1$ and $\mathcal{DPK}^1$ can be determined instantly.

Example 1.

$$\mathcal{PK}^1(G) = \begin{cases} 0, & \text{if } G = K_0 \text{ or a complete graph together with an isolated vertex,} \\ 1, & \text{if } G = K_t, t \geq 2, \\ \infty, & \text{else.} \end{cases}$$

Example 2.

$$\mathcal{DPK}^1(G) = \begin{cases} 0, & \text{if } G = K_0, K_1, \text{ or a complete graph together with two isolated vertices,} \\ 1, & \text{if } G \text{ is obtained from } K_t \text{ by adding an isolated vertex, } t \geq 2, \\ 2, & \text{if } G = K_t, t \geq 2, \\ \infty, & \text{else.} \end{cases}$$

Since posets of dimension two are more tractable than higher-dimensional posets, it is natural to expect $\mathcal{DPK}^2$ and $\mathcal{PK}^2$ to be easier to study than $\mathcal{DPK}^d$ and $\mathcal{PK}^d$ for $d > 2$, though not as easy as $\mathcal{DPK}^1$ and $\mathcal{PK}^1$.

Cho and Kim [8] initiated the study of $\mathcal{PK}^2$ while Kim, Kim and Rho [33] made interesting observations towards $\mathcal{DPK}^2$. Cho and Kim used the name *doubly partial order competition number* for what we call the dimension-2 poset competition number and they mentioned that "It is interesting to find the doubly partial order competition number of various interval graphs" [8].

The remainder of this paper is structured as follows. Section 2 summaries briefly the background of this line of research in food web study, which partly supports the legitimateness of our research here. Section 3 is concerned with $\mathcal{PK}$ and $\mathcal{DPK}$,

including interpretations of $\mathscr{P}\mathscr{K}$ and $\mathscr{D}\mathscr{P}\mathscr{K}$ in terms of intersection numbers (Theorems 8 and 9). We proceed to derive explicit formula for $\mathscr{P}\mathscr{K}^2$ in Section 4 (Theorem 18), which answers the above-mentioned open problem posed by Cho and Kim [8]. Because $\mathscr{D}\mathscr{P}\mathscr{K}^2$ looks harder to understand than $\mathscr{P}\mathscr{K}^2$, we just content ourselves in Section 5 with a proof that $\mathscr{D}\mathscr{P}\mathscr{K}^2(G)$ can be finite only if G is a trapezoid graph (Theorem 26), generalizing a result of Kim, Kim and Rho (Corollary 32).

2. Food web and intervality

A *food web* of an ecological community is a digraph with a vertex for each species in the community and $\overrightarrow{AB}$ is an arc if and only if A preys on B [43]. It is reasonable to assume that a food web is an acyclic digraph. Recent studies have shown that food webs of a wide range of ecological communities share a remarkable list of patterns [5,32,50,62]. In the NATURE paper [50], the term predator overlap graph is used for competition graph and the term prey overlap graph is used for the competition graph of the reversal of the given digraph and the edge set of the double competition graph is nothing but the intersection of the edge sets of the predator overlap graph and the prey overlap graph. These concepts are advocated by Cohen as a conceptual lens, among many others, through which to extract some pattern of the food webs [10,11,13,12,50]. Before the original finding of Cohen [10] regarding food webs and competition graphs can be stated, we need some definitions.

The *complement* of a graph $G = (V, E)$ is the graph $G^c = (V, \binom{V(G)}{2} \setminus E)$. The underlying graph of a poset is called the *comparability graph* while the complement of a comparability graph is the *incomparability graph*.

For any $n \geq 0$, the *path* P_n is the graph with vertex set $\{a_0, a_1, \ldots, a_n\}$ and edge set $\{a_i a_{i+1} : 0 \leq i \leq n - 1\}$, for which we also use the notation $[a_0 a_1 \cdots a_n]$, expressing directly the information of the positions and names of the vertices in this graph. Let P be the path $[w_0 w_1 \cdots w_m]$. An *interval* I of P is a sequence of vertices of a subpath of P, say $w_i, w_{i+1}, \ldots, w_j$, and we call $\mathbb{l}(I) = w_i$ the *left endpoint* of I and $\mathbb{r}(I) = w_j$ the *right endpoint* of I, respectively, corresponding to the given left-to-right orientation of the path P. An *interval representation* of a graph G is a path P and a set of intervals $f(v)$ of P, $v \in V(G)$, such that $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \emptyset$. A graph having an interval representation is called an *interval graph* [3,30].

Cohen collected the data of more than 30 food webs from the ecological literature and analyzed their statistical and combinatorial properties in detail. As a result, he noticed that the competition graphs of the food webs are interval graphs far more often than would be expected by chance alone and perhaps always [10,11,13,12,43]. This indicates that real food webs fall in a small subset of the mathematically possible food webs, namely acyclic digraphs.

Due to the discovery of Cohen, one of the main open problems on competition graphs is to characterize those acyclic digraphs (food webs) whose competition graphs are interval graphs [53]. This has prompted active research on the (double) competition graphs of various classes of digraphs, especially those whose (double) competition graphs are close to interval graphs. It may also be reasonable to expect that the (double) competition graph of the transitive closure of an acyclic digraph, which is a poset, will have some pattern to reflect the special structure of the digraph (poset). An interesting problem is then to characterize all posets whose (double) competition graph is a fixed graph, especially a graph close to interval graphs according to some measure. Among many other possible ways to measure the intervality, let us discuss below two ways via interval order dimension and boxicity.

An *interval order* [23,61] is a poset $(\mathscr{S}, <)$ where $\mathscr{S}$ is a family of intervals of an oriented path P and for any $I, J \in \mathscr{S}$ we have $I < J$ if and only if $\mathbb{r}(I)$ appears on the left of $\mathbb{l}(J)$ in P. The *interval order dimension* of a poset $P = (V, A)$ is the minimum number t of interval orders $(V, A_1), \ldots, (V, A_t)$ such that $A = \cap_{i=1}^{t} A_i$. Apparently, a graph is an interval graph if and only if it is the incomparability graph of an interval order (posets of interval order dimension 1). Note that Lemma 24 together with Theorem 26 will assert that the double competition graphs of posets of dimension at most two are incomparability graphs of posets of interval order dimension at most two.

Problem 3. For any positive integer d, characterize those digraphs (acyclic digraphs, posets) whose (double) competition graphs are incomparability graphs of posets of interval order dimension at most d.

A *d-grid* is a product of d paths $P_{\ell_1}, \ldots, P_{\ell_d}$, denoted $P_{\ell_1} \times \cdots \times P_{\ell_d}$, which is the graph with vertex set $\{x = (x_1, \ldots, x_d) : 0 \leq x_i \leq \ell_i, i = 1, \ldots, d\}$ and with those pairs x and y satisfying $\sum_{i=1}^{d} |x_i - y_i| = 1$ as its edge set. A *d-box* in $P_{\ell_1} \times \cdots \times P_{\ell_d}$ is a special vertex subset of the form $I_1 \times \cdots \times I_d$ where I_t is an interval in P_t, $1 \leq t \leq d$. The *boxicity* of a graph G, denoted $\mathscr{B}\mathscr{O}\mathscr{X}(G)$, is defined to be the minimum positive integer d such that G is the intersection graph of some d-boxes of a d-grid, with the only exception of complete graphs, whose boxicity is always set to be zero. Clearly, those graphs with boxicity at most one are exactly interval graphs, which have very easy recognition algorithms. In contrast, it is NP-complete to decide whether the boxicity of a graph is at most two [37].

Example 4 ([45, Theorem 7.3], [52]). It is known that $\mathscr{B}\mathscr{O}\mathscr{X}(K_{n_1,\ldots,n_t}) = t$ for any sequence of integers $n_1, \ldots, n_t > 1$.

The concept of boxicity was introduced by Roberts [52] to address niche overlap problems in ecology and was discussed further in fleet maintenance in operations research [15] and has become a concept of intensive study [6]. Cohen [53, p. 478] was interested in the minimum dimension of a niche space necessary to capture the competition relation between species. Mathematically, this is just the problem of calculating the boxicity of the corresponding competition graph. The calculations on real data suggests that this dimension (boxicity) is surprisingly low. This motivates the next question.

Problem 5. Let $b_c(d)$ ($b_{dc}(d)$) be the maximum number k such that there exists a boxicity-k graph which is the (double) competition graph of a poset of dimension at most d. How to estimate $b_c(d)$ ($b_{cd}(d)$)? Especially, are they always finite?

Note that Theorem 18 (Theorem 17) will say that $b_c(2) = 1$. We do not know whether or not $b_{dc}(2)$ is finite; though Theorem 26 and Corollary 32 will provide some relevant information on it.

3. Poset competition numbers and poset double competition numbers

An *intersection representation* $\mathcal{S}$ of a graph G is a family of sets $\{S_v : v \in V(G)\}$ such that $uv \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$. For any intersection representation $\mathcal{S}$ of G, we also say that G is the *intersection graph* of $\mathcal{S}$. The *size* of $\mathcal{S}$ refers to the cardinality of $\cup_{v \in V(G)} S_v$. For instance, the competition graph $\mathbb{C}(D)$ of a digraph D has the family of out-neighbor sets of the digraph D as an intersection representation whose size is the number of nonsource vertices of D. The minimum size of an intersection representation of G is called the *(multifamily) intersection number* of G and we reserve the notation $\mathring{i}(G)$ for it. The *competition number* of a graph G, denoted $\mathcal{K}(G)$, is the smallest nonnegative integer k such that G together with k isolated vertices is the competition graph of a digraph.

Theorem 6 (*Dutton and Brigham [17]*). *A graph G is the competition graph of an arbitrary digraph if and only if $\mathring{i}(G) \leq |V(G)|$. This means that $\mathcal{K}(G) = \max\{0, \mathring{i}(G) - |V(G)|\}$.*

A subset of $V(G)$ which induces a complete subgraph of G is called a *clique*. A clique is *maximal* if it is not contained in any other clique. A maximal clique is *trivial* if it is an isolated vertex. Let $\mathcal{C}(G)$ stand for the set of maximal cliques of G and $\mathcal{C}'(G)$ the set of nontrivial maximal cliques of G.

We think of a clique as a covering of all its two-element subsets. Thus, we can talk about an *edge clique cover* of a graph [54], namely a family of cliques each edge of the graph is covered by at least one clique from the family. The *edge clique cover number* of a graph is the minimum size of any edge clique cover of that graph.

Lemma 7 (*Erdős, Goodman and Pósa [21]*). *The intersection number of a graph equals to its edge clique cover number.*

As hinted by Lemma 7, edge clique cover has essential connection to various intersection numbers [36] and turns out to be easily understood for interval graphs and their relatives [35]. Theorem 6 tells us that the competition number is just a variant of the intersection number. Moreover, many variants of competition numbers have been revealed to be closely related to edge clique covers [17,29,41,45,48] and hence various intersection numbers. We will develop now similar results for the poset competition number and the poset double competition number.

The forthcoming result and its proof are analogous to the characterization of upper bound graphs and its proof given by McMorris and Zaslavsky [47]; see also [44, Proposition 2] or [45, Theorem 4.10].

Theorem 8. *A graph G is the competition graph of a poset if and only if its number of isolated vertices is no less than its intersection number. In other words, for a graph G with c isolated vertices, $\mathcal{PK}(G) = \max\{0, \mathring{i}(G) - c\}$.*

Proof. Suppose that G is the competition graph of a poset $P = (\{v_1, v_2, \ldots, v_n\}, <)$. Without loss of generality, assume that $v_1, \ldots, v_k$ are the minimal elements of P. Clearly, all minimal elements of P are isolated in G and thus we are reduced to proving that $k \geq \mathring{i}(G)$. For each $i \in \{1, \ldots, k\}$, let $C_i = \{v_j : v_i < v_j\}$. It is easy to check that $\{C_1, \ldots, C_k\}$ forms an edge clique cover of G. By dint of Lemma 7, we find that $k \geq \mathring{i}(G)$, as desired.

Conversely, suppose G is a graph with an edge clique cover $\{C_1, \ldots, C_k\}$ and that G has $\ell \geq k$ isolated vertices, say $\{v_1, \ldots, v_\ell\}$. Let $A = \cup_{|C_i|>1}\{\overrightarrow{uv_i} : u \in C_i\}$. It is trivial that $D = (V(G), A)$ is a poset and that $\mathbb{C}(D)$ is isomorphic to G, finishing the proof. $\square$

A *double intersection representation* $\mathcal{S}$ of a graph G is a pair of families of sets $\{S_v : v \in V(G)\}$ and $\{S'_v : v \in V(G)\}$ such that $uv \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$ and $S'_u \cap S'_v \neq \emptyset$. The *size* of the double intersection representation $\mathcal{S} = (S, S')$ is $|\cup_{v \in V(G)} S_v| + |\cup_{v \in V(G)} S'_v|$. The minimum size of a double intersection representation of G is called the *double intersection number* of G and is recorded as $d\mathring{i}(G)$.

Theorem 9. $\mathcal{DPK}(G) = \max\{0, d\mathring{i}(G) - c\}$ *where c is the number of isolated vertices of G.*

Proof. Since adding isolated vertices does not influence the value of $d\mathring{i}$, it suffices to show that a graph G is the double competition graph of a poset if and only if the number c of its isolated vertices is no less than $d\mathring{i}(G)$.

For the forward direction, we assume that $G = \mathbb{DC}(D)$ for some poset D. Suppose $\mathfrak{M}$ is the set of maximal elements of D and $\mathfrak{W}$ the set of minimal elements. Clearly, all vertices in $\mathfrak{M} \cup \mathfrak{W}$ are isolated in G and so we will be done if we can find a double intersection representation of G whose size equals the size of the symmetric difference of $\mathfrak{M}$ and $\mathfrak{W}$. For any $v \in V(G)$, put $S_v = \{u \in \mathfrak{M} : \overrightarrow{uv} \in A(D)\}$ and $S'_v = \{u \in \mathfrak{W} : \overrightarrow{vu} \in A(D)\}$. Set $\mathcal{S}$ to be the pair $\{S_v : v \in V(G)\}$ and $\{S'_v : v \in V(G)\}$. It is easy to verify that this $\mathcal{S}$ gives rise to what we need.

We continue with the backward direction. Suppose that $\mathcal{S} = (S, S')$ is a double intersection representation of G with a size no greater than the number c of isolated vertices of G. This means that there is a surjective mapping f from the set $\mathcal{I}$
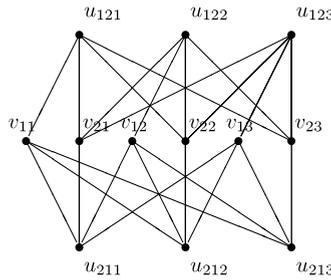
Fig. 1. The Hasse diagram of the poset $D(3, 2)$.

of isolated vertices of G to $\cup_{v \in V(G)}(S_v \cup S_v')$. By the definition of the size of the double intersection representation, we could even assume that

$$(\cup_{v \in V(G)} S_v) \cap (\cup_{v \in V(G)} S_v') = \emptyset.$$

This allows us to partition l into two parts, $\mathit{l}_1 = \{v \in \mathit{l} : f(v) \in \cup_{v \in V(G)} S_v\}$ and $\mathit{l}_2 = \{v \in \mathit{l} : f(v) \in \cup_{v \in V(G)} S_v'\}$. We define a poset D by setting $V(D) = V(G)$ and requiring $\overrightarrow{xy} \in A(D)$ if and only if either of the following three cases occurs:

$$\begin{cases} x \notin \mathit{l}_2, & y \in \mathit{l}_2 \quad \text{and} \quad f(y) \in S_x'; \\ x \in \mathit{l}_1, & y \notin \mathit{l}_1 \quad \text{and} \quad f(x) \in S_y; \\ x \in \mathit{l}_1 \quad \text{and} \quad y \in \mathit{l}_2. \end{cases} \tag{2}$$

In order to finish the proof, it remains to demonstrate $G = \mathbb{DC}(D)$, namely to check that $E(G) \subseteq E(\mathbb{DC}(D))$ and $E(G) \supseteq E(\mathbb{DC}(D))$.

Take $uv \in E(G)$. Since s is a double intersection representation of G, we can find $(x, y) \in (S_u \times S_u') \cap (S_v \times S_v')$. Due to the surjectivity of f, there are α, β such that $f(\alpha) = x$ and $f(\beta) = y$. Observe that $u, v \notin \mathit{l}$. Henceforth, according to the construction of the poset D, we have $\overrightarrow{\alpha u}, \overrightarrow{\alpha v}, \overrightarrow{u\beta}, \overrightarrow{v\beta} \in A(D)$, implying $uv \in E(\mathbb{DC}(D))$.

Consider now $uv \in E(\mathbb{DC}(D))$. This gives $\overrightarrow{\alpha u}, \overrightarrow{\alpha v}, \overrightarrow{u\beta}, \overrightarrow{v\beta} \in A(D)$ for some $\alpha, \beta \in V(D)$. In view of Eq. (2), any isolated vertex in G must be either source or sink in D. Indeed, an inspection of Eq. (2) shows that $u, v \notin \mathit{l}, \alpha \in \mathit{l}_1$ and $\beta \in \mathit{l}_2$. By now, we apply Eq. (2) again and get $f(\alpha) \in S_u \cap S_v$ and $f(\beta) \in S_u' \cap S_v'$, which guarantees $uv \in E(G)$, as desired. $\square$

It is known that computing competition numbers is an NP-hard problem [48]. We have now found some connections between intersection numbers and competition numbers. Kong and Wu [36] located some graph classes for which the intersection number is easy to calculate. It might be possible to make use of the connection here to recognize those graphs the determination of whose competition numbers is tractable.

For any graph G, Erdős, Goodman and Pósa proved that $\mathring{i}(G) \leq \lfloor \frac{|V(G)|^2}{4} \rfloor$ [21], [55, Theorem 10.2]. Let us present a variant of this classic result.

Lemma 10. *For any graph G it holds $\mathring{i}(G) \leq \left\lfloor \frac{d\mathring{i}(G)^2}{4} \right\rfloor$, or equivalently $d\mathring{i}(G) \geq \lceil \sqrt{4\mathring{i}(G)} \rceil$.*

Proof. For any double intersection representation $\mathit{s} = (S, S')$ of G, we can find that $\{S_v \times S_v' : v \in V(G)\}$ provides an intersection representation of G whose size is at most $|\cup_{v \in V(G)} S_v| \times |\cup_{v \in V(G)} S_v'| \leq \left\lfloor \frac{(|\cup_{v \in V(G)} S_v| + |\cup_{v \in V(G)} S_v'|)^2}{4} \right\rfloor$. $\square$

Problem 11. When does it hold $\mathring{i}(G) = \left\lfloor \frac{d\mathring{i}(G)^2}{4} \right\rfloor$? Equivalently, for which kind of posets D do we have $\mathring{i}(\mathbb{DC}(D)) \leq \left\lfloor \frac{c^2}{4} \right\rfloor$ where c is the number of isolated vertices of $\mathbb{DC}(D)$? Is there any parallel of Lemma 7 for the double intersection number?

Example 12. Define a poset $D(n, t)$ by setting $V(D(n, t)) = \{v_{i1}, \ldots, v_{in} : i = 1, \ldots, t\} \cup \{u_{ij1}, \ldots, u_{ijn} : i \neq j \in \{1, \ldots, t\}\}$ and $A(D(n, t)) = \bigcup_{1 \leq i < j \leq t} (\{\overrightarrow{u_{ijk}u_{ji\ell}}, \overrightarrow{u_{ijk}v_{j\ell}}, \overrightarrow{v_{i\ell}u_{jik}} : k, \ell = 1, \ldots, n\} \cup \{\overrightarrow{u_{ijk}v_{ik}}, \overrightarrow{v_{jk}u_{jik}} : k = 1, \ldots, n\})$. It can be seen that $\mathbb{DC}(D(n, t))$ is the graph obtained from $K_{\underbrace{n, n, \ldots, n}_{t \text{ components}}}$ by adding $nt(t - 1)$ isolated vertices and thus $\mathcal{DPK}(K_{\underbrace{n, n, \ldots, n}_{t \text{ components}}}) \leq nt(t - 1)$ follows. For an illustration, see Fig. 1. Note that the double competition graph of $D(3, 2)$ consists of six isolated vertices $\{u_{12k}, u_{21k} : k = 1, 2, 3\}$ and a $K_{3,3}$ whose two partite sets are $\{v_{1k} : k = 1, 2, 3\}$ and $\{v_{2k} : k = 1, 2, 3\}$.

Example 13. Define $D^+(n, 2)$ to be the poset obtained from $D(n, 2)$, which is introduced in Example 12, by adding the arcs $\overrightarrow{v_{1k}v_{2\ell}}, k, \ell = 1, \ldots, n$. It is clear that $\mathbb{DC}(D^+(n, 2))$ is the same with $\mathbb{DC}(D(n, 2))$, which is $K_{n,n}$ together with $2n$ isolated vertices. Remark 34 says that $D(n, t)$ has poset dimension at least 3 for $n, t > 1$. In contrast, we can check that the poset $D^+(n, 2)$ has dimension 2 when $n > 1$. For an illustration, see Fig. 2. Note that the dotted lines are given to mark the relative
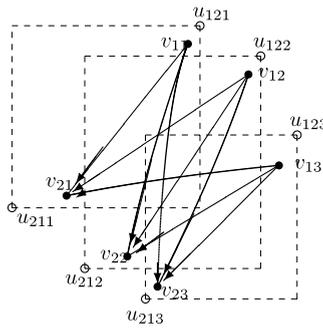
Fig. 2. The poset $D^+(3, 2)$.

positions of the points in $\mathbb{R}^2$ and we only draw those arrows from v_{1k} to $v_{2\ell}$ in order not to make the diagram too messy. This then implies that $\mathcal{DPK}(K_{n,n}) \leq \mathcal{DPK}^2(K_{n,n}) \leq 2n$. By Lemma 7 and Theorem 8 we deduce that

$$n^2 = \mathring{\i}(K_{n,n}) = \mathcal{PK}(K_{n,n}).$$

Applying Theorem 9 and Lemma 10 then yields

$$2n = \mathrm{d}\mathring{\i}(K_{n,n}) = \mathcal{DPK}(K_{n,n}).$$

This means that $\mathcal{PK}$ can take values much larger than $\mathcal{DPK}$ and that the inequality obtained in Lemma 10 is tight.

4. Dimension-2 poset competition numbers

Lemma 14 (*Opsut and Roberts [49]*). *The intersection number of an interval graph G is just $|\mathcal{C}'(G)|$.*

For any $n \geq 3$, the *cycle* C_n is the unique graph for which deleting any edge results in the graph P_{n-1}.

Theorem 15 (*Gilmore and Hoffman [27]*). *Let G be a graph. The following statements are equivalent:*

1. *The graph G is an interval graph.*
2. *The graph G is an incomparability graph and does not contain C_4 as an induced subgraph.*
3. *The maximal cliques of G can be linearly ordered such that, for each vertex v, the maximal cliques containing v occur consecutively.*

The statement that a family $\mathcal{S}$ of sets satisfies the *Helly property* means that any subfamily of pairwise intersecting sets from $\mathcal{S}$ has a nonempty intersection. The *Helly property of the real line* [60, Exercise 8.1.24] claims that any interval representation has the Helly property, namely if a set of intervals $I_1, \ldots, I_t$ of a path $P = [w_0 \cdots w_m]$ pairwise intersect then their intersection is nonempty. Indeed, this nonempty intersection is an interval with left endpoint w_ℓ and right endpoint w_r where $\ell = \max\{i : w_i = \mathbb{l}(I_j), j = 1, \ldots, t\}$ and $r = \min\{i : w_i = \mathbb{r}(I_j), j = 1, \ldots, t\}$. We mention that the general Helly theorem imparts that any set of convex sets in $\mathbb{R}^d$ has the so-called d-Helly property, being a direct generalization of the Helly property of the real line.

The *Helly multifamily intersection number* of G, denoted $\mathring{\i}_h(G)$, is the minimum size of an intersection representation $\mathcal{S}$ of G which has the Helly property.

Lemma 16 (*Kong and Wu [36]*). *The Helly multifamily intersection number of a graph G is exactly $|\mathcal{C}(G)|$.*

Cho and Kim obtained the following interesting result.

Theorem 17 (*[8, Theorem 1]*). *Competition graphs of posets of dimension at most two are interval graphs.*

The next result strengthens Theorem 17 and answers an open problem in [8].

Theorem 18. *A graph G is the competition graph of a poset of dimension at most two if and only if it is an interval graph and its number of maximal cliques is at most twice the number of its isolated vertices, the latter, by Lemmas 14 and 16, being equivalent to the assertion that $\mathring{\i}_h(G) \geq 2\mathring{\i}(G)$. In other words,*

$$\mathcal{PK}^2(G) = \begin{cases} \max\{0, 2\mathring{\i}(G) - \mathring{\i}_h(G)\}, & \text{if } G \text{ is an interval graph,} \\ \infty, & \text{otherwise.} \end{cases}$$

To present a proof of Theorem 18, we need to do some preparations.

Let S be a set of n different points

$$p_i = (x_i, y_i), \quad i = 1, \ldots, n, \tag{3}$$

in the Euclidean plane $\mathbb{R}^2$. For any $i \neq j$, write $p_i \searrow p_j$ if $x_i \leq x_j$ and $y_i \geq y_j$, and write $p_i \nearrow p_j$ if $x_i > x_j$ and $y_i > y_j$. Clearly, both $(S, \searrow)$ and $(S, \nearrow)$ are posets. Furthermore, we note that exactly one of $p_i \searrow p_j$, $p_i \nearrow p_j$, $p_j \searrow p_i$, and $p_j \nearrow p_i$ holds. This can be summarized as:

Observation 19. The underlying graph of $(S, \searrow)$ and the underlying graph of $(S, \nearrow)$ are complements of each other.

Observation 19 says that an antichain in $(S, \nearrow)$ is a chain in $(S, \searrow)$ and vice versa. In particular, we have:

Observation 20. The set of isolated vertices of the digraph $D = (S, \nearrow)$ can be enumerated as $\alpha_1 \searrow \alpha_2 \searrow \cdots \searrow \alpha_{\ell-1}$. Take two new points $\alpha_0, \alpha_\ell \in \mathbb{R}^2$ such that $\alpha_0 \searrow v \searrow \alpha_\ell$ is valid for any $v \in S$. For any vertex $v \in S \setminus \{\alpha_1, \ldots, \alpha_{\ell-1}\}$, there must be a unique $0 \leq t \leq \ell - 1$ such that $\alpha_t \searrow v \searrow \alpha_{t+1}$ and we record this by saying that v has *stage number* $\mathfrak{s}(v) = t$. We also put $\mathfrak{s}(\alpha_t) = t$. The set of sources of $(S, \nearrow)$ can be enumerated as

$$M_{11}, \ldots, M_{1r_1}, \alpha_1, M_{21}, \ldots, M_{2r_2}, \alpha_2, \ldots, \alpha_{\ell-1}, M_{\ell 1}, \ldots, M_{\ell r_\ell}$$

and the set of sinks of $(S, \nearrow)$ can be enumerated as

$$m_{11}, \ldots, m_{1s_1}, \alpha_1, m_{21}, \ldots, m_{2s_2}, \alpha_2, \ldots, \alpha_{\ell-1}, m_{\ell 1}, \ldots, m_{\ell s_\ell},$$

both of which form a chain in $(S, \searrow)$ along the above ordering.

We will also need the following.

Observation 21. Suppose that $p_1 \searrow p_2 \searrow p_3$. If $p \nearrow p_1$, $p \nearrow p_3$, then we have $p \nearrow p_2$; if $p_1 \nearrow p$, $p_3 \nearrow p$, then we have $p_2 \nearrow p$.

For any $p \in S$, set $NE(p) = \{p' \in S : p' \nearrow p\}$ and $SW(p) = \{p' \in S : p \nearrow p'\}$.

Proof of Theorem 18. We begin with the forward direction. Let $S = \{p_i : i = 1, \ldots, n\}$ be a set of n points of $\mathbb{R}^2$. Consider $D = (S, \nearrow)$, a poset of dimension at most two. Use $\mathcal{M}(D)$ for the set of minimal elements of D. By Observation 20, $\mathcal{M}(D)$ is a chain in $(S, \searrow)$, say $p_1 \searrow p_2 \searrow \cdots \searrow p_m$, without loss of generality. The combination of the following two facts implies that $\mathbb{C}(D)$ is an interval graph:

- It follows from Observation 21 that for any $p \in S$, $I_p = SW(p) \cap \mathcal{M}(D)$ is an interval of $\mathcal{M}(D)$ ordered as $p_1, \ldots, p_m$.
- By definition, $SW(p) \cap SW(q) \neq \emptyset$ if and only if $pq \in E(\mathbb{C}(D))$. Thus, by the transitivity of a poset, we get that $I_p \cap I_q \neq \emptyset$ if and only if $pq \in E(\mathbb{C}(D))$.

For any nontrivial clique C of $\mathbb{C}(D)$, the Helly property for real line gives that $I_C = \cap_{v \in C} I_v$ is a nonempty interval. It is obvious that $I_{C_1} \cap I_{C_2} = \emptyset$ for any $C_1 \neq C_2 \in \mathcal{C}'(\mathbb{C}(D))$. Thus there is an injective mapping from $\mathcal{C}'(\mathbb{C}(D))$ to the set of isolated vertices of $\mathbb{C}(D)$ and hence the forward direction is established.

Conversely, take any interval graph G with $\mathring{\mathfrak{i}} = \mathring{\mathfrak{i}}(G)$ nontrivial maximal cliques and $\mathring{\mathfrak{i}}_h - \mathring{\mathfrak{i}} = \mathring{\mathfrak{i}}_h(G) - \mathring{\mathfrak{i}}(G)$ isolated vertices. According to Theorem 15, we suppose that the maximal cliques of G are enumerated as $C_1, \ldots, C_{\mathring{\mathfrak{i}}}, C_{\mathring{\mathfrak{i}}+1}, \ldots, C_{\mathring{\mathfrak{i}}_h}$, where for each vertex v of G there is an interval $f(v)$ of $[1\, 2\, \ldots\, \mathring{\mathfrak{i}}_h]$ such that the set of maximal cliques containing v are C_t, $t \in f(v)$. Surely, we could assume that $V(G) = \{v_1, \ldots, v_n\}$ and $C_t = \{v_{t-\mathring{\mathfrak{i}}}\}$, $t = \mathring{\mathfrak{i}} + 1, \ldots, \mathring{\mathfrak{i}}_h$, are all those singleton sets in $\mathcal{C}(G)$. Put

$$p_t = \begin{cases} (t, -t) \in \mathbb{R}^2, & \text{if } t = 1, \ldots, \mathring{\mathfrak{i}}_h - \mathring{\mathfrak{i}}, \\ \left(\mathsf{r}(f(v_t)) + \dfrac{1}{2}, -\mathsf{l}\left(f(v_t)\right) + \dfrac{1}{2}\right) \in \mathbb{R}^2, & \text{if } \mathring{\mathfrak{i}}_h - \mathring{\mathfrak{i}} < t \leq n. \end{cases}$$

This then gives us a poset of dimension at most two, namely $D = (S, \nearrow)$, where $S = \{p_1, \ldots, p_n\}$. Since $\mathring{\mathfrak{i}}_h \geq 2\mathring{\mathfrak{i}}$, we have $\mathring{\mathfrak{i}}_h - \mathring{\mathfrak{i}} \geq \mathring{\mathfrak{i}}$ and so $\{(t, -t) : 1 \leq t \leq \mathring{\mathfrak{i}}\} \subseteq \{p_t : 1 \leq t \leq \mathring{\mathfrak{i}}_h - \mathring{\mathfrak{i}}\} \subseteq S$. By identifying p_t with v_t for each t, we can see readily that $\mathbb{C}(D)$ is just G and this is the proof. $\quad\square$

As consequences of Theorems 8 and 9, we obtain that both $\mathscr{P}\mathscr{K}$ and $\mathscr{D}\mathscr{P}\mathscr{K}$ take finite values and so the two decreasing sequences in Eq. (1) are eventually finite. Here is a natural question.

Problem 22. For a graph G, how to determine the minimum positive integer t such that $\mathscr{P}\mathscr{K}^t(G) < \infty$? Characterize those graphs for which such t is some given number. For $t = 1, 2$, an answer to the latter question can be gleaned from Example 1 together with Theorem 18. The same questions can be formulated for double competition numbers.

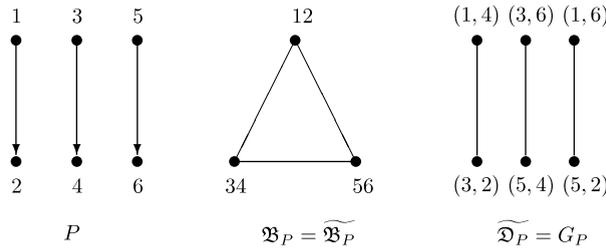**Fig. 3.** $\mathfrak{B}_P$ is not bipartite and $\mathfrak{D}_P$ is bipartite.
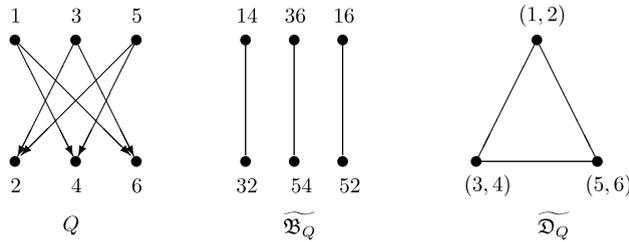

Fig. 4. $\mathfrak{B}_Q$ is bipartite and $\mathfrak{D}_Q$ is not bipartite.

5. Double competition graphs of posets of dimension at most two

Returning to Examples 4 and 13, we see that for $n > 1$, the poset $D^+(n, 2)$ has dimension two and $\mathbb{DC}(D^+(n, 2))$ has boxicity two. Therefore, we could not expect that Theorem 17 still holds when competition graph is replaced by double competition graph there. This section intends to work towards some counterpart of Theorem 17 for double competition graphs of posets of dimension at most two.

The poset $\mathbf{2} + \mathbf{2}$ is the poset whose underlying graph is the disjoint union of two P_1. Equivalently, it is the poset whose incomparability graph is C_4. If we have $x < y$ and $z < w$ in the $\mathbf{2} + \mathbf{2}$ on the vertex set $\{x, y, z, w\}$, the two *diagonals* of this $\mathbf{2} + \mathbf{2}$ are the two ordered pairs (x, w) and (z, y). For any poset P, the *diagonal graph* $\mathfrak{D}_P$ is the graph with the set of ordered pairs of incomparable elements of P as vertex set and there is an edge consisting of (x, w) and (z, y) if and only if they are the two diagonals of a common $\mathbf{2} + \mathbf{2}$ induced by $\{x, y, z, w\} \in \binom{V(P)}{4}$ in P. Fishburn [24] discovered that a poset P is an interval order if and only if $\mathfrak{D}_P$ contains no edges. It is not hard to see that this result is a reformulation of the equivalence between statements 1 and 2 in Theorem 15. Another graph which reflects the geometrical structure of those $\mathbf{2} + \mathbf{2}$ in a poset P is the graph $\mathfrak{B}_P$ which has $E(G_P)$ as vertex set and there is an edge between xy and zw in $\mathfrak{B}_P$ if and only if x, y, z, w induce a $\mathbf{2} + \mathbf{2}$ in P. Observe that $\mathfrak{B}_P$ is totally determined by G_P. Let $\widetilde{\mathfrak{B}_P}$, respectively, $\widetilde{\mathfrak{D}_P}$, be the graph obtained from $\mathfrak{B}_P$, respectively, $\mathfrak{D}_P$, by deleting all isolated vertices.

Example 23. Let V_1 and V_2 be two disjoint sets and let $V_1 \times V_2$ be the disjoint union of A_1 and A_2. Let P be the poset $(V_1 \cup V_2, A_1)$ and Q be the poset $(V_1 \cup V_2, A_2)$. It is not hard to check that $\widetilde{\mathfrak{B}_P} = \widetilde{\mathfrak{D}_Q}$ and $\widetilde{\mathfrak{B}_Q} = \widetilde{\mathfrak{D}_P}$. For an illustration, see Figs. 3 and 4.

A *trapezoid graph* is a graph having a *trapezoid representation*, namely the intersection graph of a family of trapezoids whose two parallel sides lie along two given horizontal lines. This concept was introduced by Corneil and Kamula [14], and independently by Dagan, Golumbic and Pinter [16], as a generalization of interval graphs. A poset of interval order dimension at most two is also called a *trapezoid order*. This name comes from its geometric interpretation in terms of a set of trapezoids [4, Definition 6.5.2]. It is easy to prove but important to realize, as Dagan, Golumbic and Pinter did [16], that the following holds.

Lemma 24 ([28, Remark 5.19]). *The set of trapezoid graphs is exactly the set of incomparability graphs of posets of interval order dimension at most two, namely trapezoid orders.*

Moreover, Cogis [9] deduced the following excellent characterization.

Theorem 25 ([4, Lemma 6.5.1][22, Lemma 1]). *A poset P is a trapezoid order if and only if its diagonal graph $\mathfrak{D}_P$ is a bipartite graph.*

Here comes our main result, which may be still far away from a complete understanding of the double competition graphs of posets of dimension no bigger than two.

Theorem 26. *Let $D = (S, \lessdot)$ be a poset of dimension at most 2. Then $\mathbb{DC}(D)$ is the incomparability graph of a poset P satisfying that both $\mathfrak{B}_P$ and $\mathfrak{D}_P$ are bipartite. Moreover, $\mathbb{DC}(D)$ is a trapezoid graph.*

The next problem is motivated by Example 23, Theorems 25 and 26.

Problem 27. Is there any good intersection representation for the incomparability graph of a poset P with $\mathfrak{B}_P$ being bipartite? Is there any good relationship between $\mathfrak{B}_P$ and $\mathfrak{D}_P$ for posets P? For which posets P do we have $\mathfrak{B}_P = \mathfrak{D}_P$?

Before we get to a proof of Theorem 26, we have to gone through several observations. Let us start with three simple facts, the latter two were already explicitly recorded by Kim, Kim and Rho in [33].

Observation 28. Every source and every sink of a digraph D becomes an isolated vertex in $\mathbb{DC}(D)$.

Lemma 29 ([33, Lemma 1]). *Let $D = (S, \swarrow)$ be a poset of dimension at most 2. If both p and q are ordinary vertices of D and $\overrightarrow{pq} \in A(D)$, then $pq \in E(\mathbb{DC}(D))$.*

Proof. Since p is not a source, we can take a p' with $\overrightarrow{p'p} \in A(D)$; since q is not a sink, there is a q' such that $\overrightarrow{qq'} \in A(D)$. We then have $p' \swarrow p \swarrow q \swarrow q'$, which gives $pq \in E(\mathbb{DC}(D))$, as wanted.   □

Lemma 30 ([33, Lemma 2]). *Let $D = (S, \swarrow)$ be a poset of dimension at most two. Suppose p, q, r are vertices of D satisfying $p \searrow q \searrow r$. If $pr \in E(\mathbb{DC}(D))$, then $pq, qr \in E(\mathbb{DC}(D))$.*

Proof. This is a consequence of Observation 21.   □

We are ready to present a crucial observation.

Lemma 31. *Let $D = (S, \swarrow)$ be a poset of dimension at most 2. Then $\mathbb{DC}(D)$ is an incomparability graph.*

Proof. Let us appeal to Observation 20 and follow the notation introduced there. Use the shorthand G for $\mathbb{DC}(D)$. For any edge $pq \in E(G^c)$, we orient it to be $\overrightarrow{pq}$ if and only if one of the followings holds:

(1) $\mathfrak{s}(p) < \mathfrak{s}(q)$.
(2) Both p and q are sources and $p \searrow q$.
(3) Both p and q are sinks and $p \searrow q$.
(4) $\mathfrak{s}(p) = \mathfrak{s}(q)$, p is a source and q is not a source.
(5) $\mathfrak{s}(p) = \mathfrak{s}(q)$, p is ordinary, $q \neq \alpha_{\mathfrak{s}(q)}$, q is a sink.
(6) $\mathfrak{s}(p) = \mathfrak{s}(q)$, both p and q are ordinary vertices, and $p \searrow q$.

By giving the above orientation rules, we have disposed of all possible cases for the edge $pq \in E(G^c)$. It is noteworthy that when p and q are two ordinary vertices with $\mathfrak{s}(p) = \mathfrak{s}(q)$, we can assume that $p \searrow q$, which is the case we treat with orientation rule (6), due to the combination of Observation 19 and Lemma 29.

To complete the proof, we need to show that the digraph P obtained from G^c by assigning the above orientation is a poset. It is clear that P has no loops and so it suffices to derive from the existence of the arcs $\overrightarrow{pq}, \overrightarrow{qr} \in A(P)$ that $\overrightarrow{pr} \in A(P)$. The verification is done in several cases.

CASE 1: $\mathfrak{s}(p) \neq \mathfrak{s}(r)$. By orientation rule (1), we must have $\mathfrak{s}(p) < \mathfrak{s}(r)$ and only need to establish $pr \in E(G^c)$. It follows from Observation 28 that $p\alpha_{\mathfrak{s}(r)} \in E(G^c)$. Hence, we are already done in the case that $r = \alpha_{\mathfrak{s}(r)}$. In the remaining case, we have $p \searrow \alpha_{\mathfrak{s}(r)} \searrow r$. Hence, Lemma 30 together with the fact that $p\alpha_{\mathfrak{s}(r)} \in E(G^c)$ gives $pr \in E(G^c)$, as desired.

CASE 2: $\mathfrak{s}(p) = \mathfrak{s}(r)$.

SUBCASE 2.1: At least one of p and r is not an ordinary vertex. Observation 28 allows us to assert that $pr \in E(G^c)$ and so our object of study is the orientation of pr in P.

We claim that if r is a source then p must be a source and if p is a sink then r has to be a sink. Indeed, orientation rule (4) or (5) will be violated if this were not the case. The above observation suggests that we need to consider the following subcases:

(2.1.1) The vertex p is a source and r is not a source.
(2.1.2) Both p and r are sources.
(2.1.3) Both p and r are sinks.
(2.1.4) The vertex p is ordinary while r is a sink.

For (2.1.1), orientation rule (4) shows that $\overrightarrow{pr} \in A(P)$.

For (2.1.2), we know from orientation rules (2) and (4) that q is also a source and $p \searrow q \searrow r$. Now, by orientation rule (2) we get $\overrightarrow{pr} \in A(P)$.

For (2.1.3), it follows from orientation rules (3), (4) and (5) that q is also a sink and $p \searrow q \searrow r$. Henceforth, orientation rule (3) gives what we want.

Finally, taking into account orientation rule (5), we obtain the result for (2.1.4).

SUBCASE 2.2: Both p and r are ordinary vertices, which then implies that q is also ordinary due to the orientation rules (4) and (5). By orientation rule (6) and Lemma 29, we conclude that $p \searrow q \searrow r$ and so $\overrightarrow{pr} \in A(P)$ provided we have $pr \in E(G^c)$. But how to prove $pr \in E(G^c)$? What comes to the rescue is Lemma 30. Suppose to the contrary that $pr \in E(G)$. Lemma 30 guarantees that $pq, qr \in E(G)$, contradicting our assumption that $\overrightarrow{pq}, \overrightarrow{qr} \in A(P)$. This is the end of the proof.   □

Many results here are developed from the fundamental observations of Kim, Kim and Rho in [33]. It is thus no wonder that all results in [33] follow from Lemma 31 and some basic facts in intersection graph theory. We only give one such example below.

Corollary 32 (*Kim, Kim and Rho [33, Theorem 6]*). *Let G be the double competition graph of a poset of dimension at most two. If G does not have C_4 as an induced subgraph, then G is an interval graph.*

Proof. This follows readily from Theorem 15 and Lemma 31. □

To go a step further from Lemma 31 to Theorem 26, we require one more preparatory result. It can be essentially extracted from the proof of [33, Theorem 8] and is quite useful in [38,39].

Lemma 33. *Let $D = (S, \nearrow)$ be a poset of dimension at most 2. Suppose $(uvxw) = C_4$ is an induced subgraph of $\mathbb{DC}(D)$. Then after a suitable relabeling of the four vertices $u, v, x, w \in \mathbb{R}^2$ which induces an automorphism of the corresponding induced cycle, we can have $u \nearrow v, x \nearrow w, v \searrow w, u \searrow x$.*

Proof. It is straightforward from Lemma 29 that we can assume $v \searrow w$ and $u \searrow x$. Noting that $vx \in E(\mathbb{DC}(D))$, $ux \notin E(\mathbb{DC}(D))$, and $u \searrow x$, we deduce from Lemma 30 that it cannot occur $v \searrow u$. Applying similar argument yields that $u \searrow v, x \searrow w$ and $w \searrow x$ are also impossible. Therefore, we know that one of $u \nearrow v$ and $v \nearrow u$ and one of $x \nearrow w$ and $w \nearrow x$ must hold. Without loss of generality, we suppose that $u \nearrow v$. What remains to do is to rule out the possibility of $w \nearrow x$. Since $uw, vx \in E(\mathbb{DC}(D))$, we could find in the poset D a common strict lower bound L_{vx} of v and x as well as a common strict upper bound U_{uw} of u and w. Meanwhile, we conclude from $u \nearrow v$ that U_{uw} is a common strict upper bound of v and w. Assume now, by way of contradiction, that $w \nearrow x$. It then follows that L_{vx} is a common strict lower bound of v and w, violating the assumption that $vw \notin E(\mathbb{DC}(D))$. This contradiction completes the proof of the lemma. □

Remark 34. Recall the poset $D(n, t)$ as introduced in Example 12. Take $n, t > 1$. It is obvious that $(v_{11}v_{21}v_{12}v_{22})$ is an induced 4-cycle in $\mathbb{DC}(D(n, t))$. If $D(n, t)$ has poset dimension no greater than 2, Lemma 33 says that there are at least four arcs in $D(n, t)$ among $v_{11}, v_{21}, v_{12}, v_{22}$, contradicting with the structure of $D(n, t)$. This means that $D(n, t)$ has poset dimension at least 3.

Proof of Theorem 26. We call $p_i p_j \in \binom{S}{2}$ a red pair provided p_i and p_j have a common strict upper bound in the poset D and call it a green pair whenever p_i and p_j have a common strict lower bound in the poset D.

Take the poset P described in the proof of Lemma 31. Recall that we already know there that the incomparability graph of P is just $\mathbb{DC}(D)$. To test the bipartiteness of $\mathfrak{B}_P$, our task is to check the next two claims:

- If $p_i p_j \in V(\mathfrak{B}_P)$ is not isolated, then it is either red or green, but not both;
- If $p_i p_j$ and $p_k p_\ell$ are connected by an edge in $\mathfrak{B}_P$, then one of them is red and the other is green.

These two can be read from Lemma 33 simultaneously. Indeed, by Lemma 33, we can assume that $p_i \nearrow p_k, p_j \nearrow p_\ell$. Since $p_i p_\ell$ is green, we obtain from $p_j \nearrow p_\ell$ that $p_i p_j$ is also green. From $p_i p_j \notin E(\mathbb{DC}(D))$ we educe that $p_i p_j$ cannot be red. Analogously, $p_k p_j$ is red along with $p_j \nearrow p_\ell$ implies that $p_k p_\ell$ is red. Furthermore, $p_k p_\ell \notin E(\mathbb{DC}(D))$ guarantees that $p_k p_\ell$ is not green. This completes the proof that $\mathfrak{B}_P$ is bipartite.

Because of Lemma 24 and Theorem 25, it remains to establish the bipartiteness of $\mathfrak{D}_P$. Consider a $\mathbf{2 + 2}$ in P whose two arcs are, say $\overrightarrow{p_i p_j}$ and $\overrightarrow{p_k p_\ell}$. From Lemma 33 we could assume without loss of generality that the following hold:

(a) $p_k \searrow p_\ell$;
(b) $p_i \nearrow p_k$;
(c) $p_i \searrow p_j$;
(d) p_j and p_k are incomparable in P.

We follow Eq. (3) in denoting the two coordinates of a point in $\mathbb{R}^2$. The combination of (a) and (b) gives $y_i > y_\ell$; the combination of (b) and (c) implies that $x_j > x_k$ and hence we see from (d) that $y_j > y_k$. Observe that the diagonals of this $\mathbf{2 + 2}$ is $\overrightarrow{p_i p_\ell}$ with $y_i > y_\ell$ and $\overrightarrow{p_k p_j}$ with $y_j > y_k$. It is now clear that for any edge of the diagonal graph, say $\{\overrightarrow{p_i p_\ell}, \overrightarrow{p_k p_j}\}$, the sign of $y_i - y_\ell$ and $y_k - y_j$ are opposite to each other and hence $\mathfrak{D}_P$ must be bipartite, as wanted. □

Theorem 26 says that to be the incomparability graph of a poset P satisfying that both $\mathfrak{B}_P$ and $\mathfrak{D}_P$ are bipartite is a necessary condition for a graph to be a double competition graph of a poset of dimension at most two. Let us mention that we [38] already found two forbidden subgraphs for double competition graphs of posets of dimension at most two, both of which being incomparability graphs of posets P satisfying that both $\mathfrak{B}_P$ and $\mathfrak{D}_P$ are bipartite.

Example 35. The *Roberts graph* $\mathscr{R}_n$ on $2n$ vertices is the graph obtained from K_{2n} by removing a perfect matching [6,52]. Roberts [52] pointed out that the boxicity of $\mathscr{R}_n$ is at least n. It is quite simple to find a trapezoid representation for Roberts graphs and so we know that trapezoid graphs can have arbitrarily high boxicity.
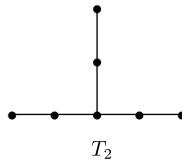
Fig. 5. The graph T_2.

Example 36. For $n \geq 1$, set Σ_n to be the poset whose underlying graph is the disjoint union of nP_1. Clearly, the Roberts graph $\mathscr{R}_n$, introduced in Example 35 and being a trapezoid graph, is just $G_{\Sigma_n}^c$. Similar to the case depicted in Fig. 1, we find that $\mathfrak{B}_{\Sigma_n} = K_n$ and that $\widetilde{\mathfrak{D}}_{\Sigma_n}$ is a disjoint union of $\binom{n}{2} P_1$. Note that for any two posets P and Q with $G_P^c = G_Q^c$, we must have $\mathfrak{B}_P = \mathfrak{B}_Q$. Thus, according to Theorem 26, for any $n \geq 3$ we know that $\mathscr{R}_n$, though being a trapezoid graph, cannot become a vertex induced subgraph of the double competition graph of any poset of dimension at most two.

For any graph G, set $\mathfrak{B}_G$ to be the graph with vertex set $E(G)$ and there is an edge connecting xy and zw in $\mathfrak{B}_G$ if and only if x, y, z, w induce a $2P_1$ in G. We use the notation $\widetilde{\mathfrak{B}}_G$ for the graph obtained from $\mathfrak{B}_G$ by removing all its isolated vertices. It is worth pointing out that if $G = G_P$ for a poset P, then we have $\mathfrak{B}_P \cong \mathfrak{B}_G$ and $\widetilde{\mathfrak{B}}_P \cong \widetilde{\mathfrak{B}}_G$.

Corollary 37. *If G is a double competition graph of a poset of dimension at most two, then $\mathfrak{B}_{G^c}$ is a bipartite graph.*

Proof. This follows directly from Theorem 26 and the remark before this corollary. □

For any interval I on the real line, we denote its length by $|I|$. A graph is a *bounded tolerance graph* if each vertex $v \in V(G)$ can be assigned a closed interval I_v in the real line and a tolerance t_v satisfying $0 \leq t_v \in |I_v|$ so that $uv \in E(G)$ if and only if $|I_u \cap I_v| \geq \min(t_u, t_v)$ [28, p. 5].

Example 38. It is known that all bounded tolerance graphs are trapezoid graphs [28, Theorem 2.9]. We now demonstrate that there are bounded tolerance graphs which are not double competition graphs. Let T_2 be the graph depicted in Fig. 5. Applying [28, Theorem 3.7], we can find that the complement of T_2, namely T_2^c, is a bounded tolerance graph. On the other hand, it is easy to check that $\widetilde{\mathfrak{B}}_{T_2} = K_3$. In view of Corollary 37, T_2^c cannot be a vertex induced subgraph of the double competition graph of any poset of dimension at most two.

Theorem 26 along with Examples 36 and 38 suggests the next problem.

Problem 39. Determine those trapezoid graphs which arise from posets of dimension at most two by taking double competition graphs. It is well known that many proper subclasses of tolerance graphs are proper subclasses of trapezoid graphs [28]. Thus, especially, what is the relationship between the class of double competition graphs of posets of dimension at most two and various subclasses of tolerance graphs? A partial solution to this problem will be reported in [39].

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